Designing Dual Modeling Task Sequences to Build Functional Analysis Learning Trajectories for Engineering and Mathematical Sciences Education

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***Abstract*—This research-to-practice paper investigated the development of Functional Analysis Learning Trajectories (FALT) within the undergraduate courses for engineers including Calculus, Differential Equations, Communications Theory, Control Systems and Electromagnetism. Focusing on the local instructional practices, manifestations of functional analysis ideas are tracked along these courses. Corresponding concept maps are collaboratively within and across courses to portray the relevant connections for engineering mathematics education. These trajectories help engineering and mathematics instructors to collectively design, implement, refine task sequences to improve students' preparedness for upper-level math-heavy courses in engineering, connecting mathematics to their disciplines. Here we utilized collaborative concept mapping as a research heuristics to build curricular innovations with functional analysis learning trajectories across courses, hierarchically arranging, integrating, and conceptually connecting instructional tasks. Through sequences of dual stance learning tasks, students are given opportunities to take multiple stances in a learning task from the perspectives of engineers and mathematical scientists. A higher stance on mathematics was supported to be developed by comparing and connecting alternative disciplinary perspectives and practices with the dual modeling tasks and reflecting on the cross-cutting ideas along the learning trajectory. Here we present the collaborative design and analysis of concept maps along functional analysis learning trajectories for undergraduates. This research builds an interdisciplinary scholarship of teaching/learning mathematics across disciplines among engineering and math faculty. This work helps foster reflection and collaboration on their teaching practices to design and implement instructional tasks to build coherent mathematical perspectives across disciplines. It exemplifies how to design a research-based practices to build cross-curricular innovations. Concept maps within courses are presented and discussed here to build mathematical connections and functional analysis learning trajectories for engineering and applied mathematics education.**

***Index Terms*—Engineering Mathematics Education, Mathematical Modeling, Interdisciplinary Learning Trajectories, Collaborative Concept Maps, Fourier Analysis**

I. Introduction

Functional analysis is a well-developed field of mathematics providing a unifying perspective in solving problems in applied sciences and engineering [1]. Here is a sample of subjects with strong connections to Functional analysis: differential equations, partial differential equations, operator theory, complex analysis, representation theory, heat flow, signal processing, image processing, control theory, electromagnetic theory, and acoustics. This sample indicates how relevant functional analysis perspectives are for pure, applied and engineering mathematics [2], [3]. Many techniques in signal processing such as Fourier methods, spectrum estimation, filter design, wavelet analysis developed from collaborative work among mathematicians and engineers who used results from functional analysis to formulate problems in their disciplines [4], [5].

Due to the gaps among engineering and mathematics courses, the development and transfer of such advanced interdisciplinary mathematics perspectives are often inadequately supported and integrated along the required mathematics and engineering courses for undergraduates [6]–[8]. Fourier methods have found a significant place in engineering mathematics education programs around the world [9]. This work builds on the earlier work on Fourier analysis learning trajectory across Trigonometry, Linear algebra, Signal processing. Learning trajectories are constructed to articulate conceptual elements and connections to support interdisciplinary collaborations to design and sequence instructional activities for integrated learning of mathematics for engineering education. There is a need to identify and map learning opportunities to build functional analysis ideas across the relevant courses for undergraduate in engineering and applied mathematics education.

Mathematical modeling is an essential practice common for engineers and in interdisciplinary mathematics learning [10]–[12], providing a common ground to help students build connections and gain a higher stance to see how mathematical ideas are connected across courses and disciplines along a learning trajectory. To support the development learning trajectories in engineering and mathematics courses, we are building on research in dual modeling cycle [13] and modeling task sequences [14] with multiple disciplinary stances [12].

Linking the theory and practice in engineering mathematics education requires educational research methods and curriculum design heuristics that can recognize the complexity of learning and applying advanced mathematical content, such as Fourier transforms, for engineering and applied mathematics education. Learning trajectories are research constructs adopted here to inform, guide, and build research-based collaborations among faculty to innovate instructional practices for engineering mathematics education [15]. Collaborative concept mapping is a research heuristics to build curricular innovations hierarchically arranged, integrated, and conceptually driven [16]–[18]. It facilitates a meaningful discourse between engineering and mathematics faculty in designing learning activities helping students as guidelines to support their learning along learning trajectories [19].

II. Problem and Background

Learning trajectories provide an integrated approach to develop mathematical knowledge for teaching engineering mathematics through design and analysis of mathematical tasks for engineering and applied mathematics students. These trajectories are designed to improve teaching and learning of mathematics for engineers through applied mathematics, guiding students' learning experience; connecting, anticipating, attending to learning discourse; and providing assessments to gauge their progressions. While there are examples of Learning Trajectories in the research literature [9], there is a gap in research in the development of Fourier and Functional Analysis thinking in undergraduate mathematics education for engineers and applied mathematics students. We examine learning trajectories for research-oriented teaching and learning of functional analysis, as it builds across aligned mathematics and engineering courses. Fourier methods and Laplace transforms are among the topics that we discuss here as part of functional analysis learning Trajectory across Calculus series, ODE, Linear Algebra, Control Systems, and Image Processing courses. For these courses, we collaboratively worked as a group to reflect and build on our practices based on the student and instructor experience to address the gaps between mathematics and engineering concepts informing the design of learning trajectories.

III. Setting and Relevance

This study is undertaken at a midsize state university with large cohorts of engineering majors. The authors, as an interdisciplinary team of faculty with Mathematics/STEM education, applied mathematics, and engineering backgrounds have been leading a community of practice on scholarship of teaching and learning engineering mathematics education [20]. The team focuses on addressing the problems of teaching/learning mathematics for engineering and mathematical sciences with an underlying motivation towards preparing students to the advanced engineering courses, demanding students apply Fourier methods with an understanding of functional analysis.

This work builds on the earlier work by the authors focusing on Fourier Analysis learning trajectories across Trigonometry, Linear Algebra and Signal processing courses [9]. This collaborative research in engineering mathematics education aims to identify the learning trajectories of functional analysis, encompassing Fourier Methods, across courses to build mathematical modeling tasks with dual disciplinary stances from engineering and mathematics. Dual take on the mathematical and engineering stances allow an analysis of conceptual connections along the learning trajectory, closing the gap between mathematical courses and engineering courses. This approach helps to build an integrated learning of functional analysis for engineering and applied mathematics education. Drawing from mathematics education research [9], [21], [22] these trajectories allow experimentation on the teaching practices across courses facilitating the development and transfer of functional analysis thinking in engineering mathematics courses a

The design research as described in [15], [23], [24] is conducted here to study local instructional practices in developing teaching and learning of functional analysis across a series of mathematics and engineering courses taken by engineering students. Three mathematics faculty teaching Calculus II, Calculus III, ODE, and Linear Algebra worked together to articulate hypothetical learning trajectories for functional analysis from mathematical perspective. Three engineering faculty worked with mathematics faculty to develop relevant mathematical learning trajectories across the courses they teach in Communications and Control Theory, Image processing, signal processing and electromagnetism. They collaborated on reviewing and developing concept maps to identify conceptual elements and connections in learning and applying functional analysis across a sequence of mathematics and engineering courses. Here we present the collaborative interdisciplinary reflections on their practices across the courses to build the learning trajectories.

IV. Functional Analysis Learning Trajectories across Courses

Many applied mathematics or engineering problems can be posed as a differential equation which needs to be solved for a function satisfying initial conditions. The Laplacian operator Δ gives the difference between the average value of a function in the neighborhood of a point, and its value at that point. If u is the temperature, Δ explains the relative hotness or coldness of the material in the neighborhood of each point on the average, compared to the temperature at that point. The diffusion equation, $\frac{\partial u}{\partial t} = \Delta u$, models the distribution of a three dimensional heat flow, where $\Delta = u_{xx} + u_{yy} + u_{zz}$. Joseph

Fourier introduced the Fourier Series method in his treatise Théorie analytique de la chaleur, published in 1822 [25], to solve the heat equation on a rod, modeled as $u_t = \alpha u_{xx}$ where u is a function of x and t. By applying a transformation such as the Laplace or Fourier to each term of such equation, we arrive at a new equation that can be solved algebraically.

Functional analysis helps to understand the underlying infinite dimensional setting. Fourier series representation of a function comes from a function space composed of an n-dimensional vector into components with respect to an orthonormal basis for $\mathbb{R}^n$ or $\mathbb{C}^n$. This orthonormal basis is formed by e^{imx} where $m = 0, ..., n$. The basis is a function space formed by $\{\cos(2\pi Tkx), \sin(2\pi Tkx)\ \}$ where k is an integer from 0 to n. The inner product of these functions shows their orthogonality. $< e_n, e_m >= \int_0^L e_n(x)e_m(x)dx = 1$ if $m = n$, 0 otherwise.

A. Foundations of Functional Analysis in Calculus Series and ODE enhanced by PDE

In Calculus 2 and 3, students learn about integration and series including trigonometric integrals $\int \sin \mathrm{x} \cos \mathrm{x} \mathrm{dx}$ associated by Fourier Transform, and improper integrals $\int e^{-x} \sin \mathrm{x} \mathrm{dx}$, associated by Laplace Transform. While students learn about the techniques such as integration by parts, they usually do not have opportunity to have opportunity to take multiple disciplinary stances to connect mathematics into engineering. The dual modeling approach we describe here helps students see why they are doing these integration problems and how these integral becomes fundamental to build functional and Fourier analysis ideas from mathematical and engineering stances. Along the functional analysis learning trajectory, the embedded learning activities on the improper integrals $\int e^{-x} \sin \mathrm{x} \mathrm{dx}$ towards building foundation for Laplace transform. Laplace transform of a cosine function $\int_0^\infty e^{-st} \cos \mathrm{a} t dt$ is done in Calculus 2 using integration by parts to develop $\frac{s}{s^2+\mathrm{a}^2}$ without using the properties of Laplace transform. Similarly, Fourier transform of a damped cosine function is calculated by integration by parts. Fourier transform of cosine function is expressed in terms of the sum of two Dirac's delta function $1/2(\delta(f - f_0) + \delta(f + f_0))$. Students connect the Dirac Delta function as a forcing function describing an impulse such as a large voltage exerted on the electrical system at certain time. $\int_{a-\epsilon}^{a+\epsilon} \delta(t - a)dt = 1$ where $\epsilon > 0$. Laplace transform for delta function is investigated as $\int_0^\infty e^{-st}\delta(t - a)dt = e^{-as}$. The connection of δ to Heaviside step function is realized, with $\int_0^\infty \delta(t - a)dt = 0$ if $t < a$ and 1 if $t > a$.

While these integrals are common practice in Calculus II, and assessed in midterms and final exams, students need support to gain an interdisciplinary stance and see how these ideas becomes fundamental in Laplace transforms for solving engineering problems. This kind of metacognitive activity is a desired learning goal with collaborative concept maps helping instructors and students build the connections and anticipating how these ideas will help build more complex learning tasks relevant to their disciplines.

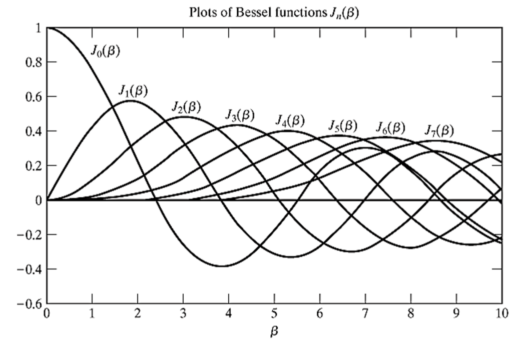

Fig. 1. The Bessel functions of different order

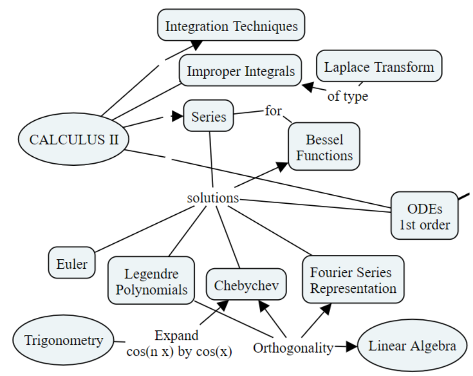

Fig. 2. Concept map for Functional Analysis Learning trajectory in Calculus II from Math and Engineering Stances with its connections

Along the FA learning trajectory, the embedded learning activities in Calculus III focused on the series representations of special functions as solutions to a class differential equations. The representative project for Calculus III were planned on the series' representations for Bessel functions (Figure 1). Bessel functions are used in modeling the behavior of a vibrating drumhead as a modeling project. This modeling project allowed students to explore the connections between Bessel functions of different orders and differential equations. The properties of Bessel polynomials as orthogonal sequence of polynomials were embedded into Linear Algebra as a part of learning trajectory in discussing orthogonality. Bessel function of order 0 defined by

$$\mathcal{J}_0(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{2^{2n}(n!)^2}$$

. $\mathcal{J}_0$ satisfies the differential equation

$$x^2\mathcal{J}_0''(x) + x\mathcal{J}_0'(x) + x^2\mathcal{J}_0(x) = 0.$$

In ODE, Bessel's differential equation is reintroduced $x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + (x^2 - \alpha^2)y = 0$ where α is a complex number representing the order of Bessel function.

The Chebyshev polynomials provide another learning opportunity to build from trigonometry the sequences of polynomials using cosine and sine functions. The Chebyshev polynomials of the first kind $T_n(x)$ are defined by

$$T_n(x) = \cos(nx)$$

The orthogonality of Chebyshev polynomials is a part of the learning trajectory in Linear algebra with inner product below $\langle f, g\rangle = \int_{-1}^{1} f(x)\, g(x)\, \frac{\mathrm{d}x}{\sqrt{1-x^2}}.$

Connections along the learning trajectory becomes more apparent building connections between Calculus II and Calculus III and Ordinary Differential Equations, Communications and Electromagnetics. While the calculus 3 is required for Electrical engineering, Calculus 2 is often the last calculus course for many engineering majors in our institution. Students miss the introduction of the partial differential equation ideas in Calculus 3 with its connections to ordinary differential equations. One of the major need for applied math/engineering students is to enhance the treatment of partial differential equations to provide students with strong foundations with the partial differential equations. The following Figure 3 represents a concept map, developed collaboratively, depicting mathematical ideas and connections to guide and support embedded learning activities connecting Calc 2, Calc 3, and ODE amplifying the connections to PDE. In a sense, these activities foster a functional analysis approach in the solution of differential (ODE, PDE) and integral equations with connections to solution spaces obtained by Fourier and Laplace methods.

An example of dual stance here is that from mathematical stance the Green's function $G(x,t)$ is the fundamental solution for the heat equation for a circle. From engineering stance, Green's function is the "impulse response" associated with the linear system "heat flow on a circle". In general, Green function is the solution of $L(G(x,t)) = \delta(x-s)$ where is a linear differential operator. Once Green functions found for each s, they can be superimposed to solve $L(u(x)) = f(x)$ with the convolution of $G * f$, given that the source is a sum of delta functions, the solution is a sum of Green's functions due to linearity. For the heat equation, the initial heat distribution $f(x)$ generates the temperature $u(x,t)$. Once the fundamental solution as heat kernel is found the solution of the original equation is obtained through convolution of the fundamental solution. In signal systems, the concept of convolution is introduced from engineering perspective. The engineering faculty in our institution identified convolutions as one of the hardest concepts for engineering students to understand without proper mathematical background.

Considering the Functional Analysis Learning Trajectory across Calculus series and ODE, we collaboratively developed the concept map represented by the Figure 3 to describe Mathematical ideas and connections to develop Functional Analysis Learning Trajectory across Calculus and Ordinary Differential as well as introducing foundational partial differential equation ideas related to functional analysis. Functional analysis idea underpins the practice of Laplace transform in the solutions

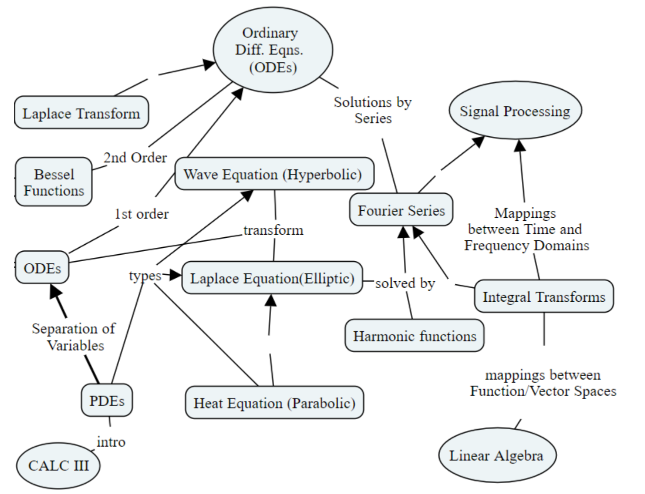

Fig. 3. Collaborative concept map supporting Functional Analysis Learning Trajectory across Calculus 3, Ordinary Differential, and PDE ideas

of differential equations. Transforming functions and equations into another class of functions by integral transforms.

The FALT in ODE includes Laplace Transform by which the differential equations are transformed into algebraic expressions with the following integration

$$\mathscr{L}(f(t)) = \int_0^\infty e^{-st} f(t)dt$$

In ODE, Bessel functions are important in modeling wave propagation. The models for the vibrating drum with partial differential equation which can be reduced to Helmoltz equation by separation of variables, for which the equation can be converted to a second order ODE with Bessel functions as solutions. This builds mathematical foundations for electromagnetics course for electrical engineers.

The Laplace equation is a second-order partial differential equation given by $\nabla^2 u = 0$ where ∇^2 is the Laplacian operator. In two dimensions, $\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}$. Students learn that Fourier series can be used to solve this Laplace equation with the given boundary conditions where $u(x,0) = f(x)$ and the rest of the rectangular boundary is zero valued: $u(x,y) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right) \sinh\left(\frac{n\pi y}{H}\right)$ where B_n are coefficients determined by the boundary condition $u(x,0) = f(x)$. These coefficients can be found using the orthogonality of sine functions. $B_n = \frac{2}{L}\int_0^L f(x) \sin\left(\frac{n\pi x}{L}\right) dx$. This is a general form of the solution, and the specific solution would depend on the function $f(x)$ defined on the boundary $y = 0$.

In ODE course, electrical engineering stance is reinforced by modeling RLC (resistor, inductor, and capacitor) circuits which provides a good context for solving ordinary differential equations by using Laplace transform. These circuits are connected in series with a voltage source $V(t)$. Students are expected to develop the differential equation governing the circuit's current$i(t)$ as $L\dfrac{d^2 i}{dt^2} + R\dfrac{di}{dt} + \dfrac{1}{C} i = V(t)$. Assuming that the circuit is subject to a step input voltage $V(t) = V_0 u(t)$, where $u(t)$ is the unit step function and V_0 is a constant voltage. Students use Laplace Transform by applying it to both sides of the equation, where $\mathcal{L}\{i'(t)\} = sI(s) - i(0)$ and for a second derivative, $\mathcal{L}\{i''(t)\} = s^2 I(s) - si(0) - i'(0)$. Assuming zero initial conditions for current and its derivative, the transformed equation becomes: $Ls^2 I(s) + RsI(s) + \frac{1}{C} I(s) = \frac{V_0}{s}$. Students then rearrange the equation to solve for the Laplace transform of the current: $I(s) = \frac{V_0}{s(Ls^2+Rs+\frac{1}{C})}$.

Handling this partial fraction decomposition is where students struggle most, which requires students to coordinate multiple set of mathematical ideas including finding the factors of polynomials in the denominator and solving a system of linear equations to identify simple fractional expressions with s. These ideas should have been mastered in the calculus to simplify $I(s)$ so that students can use inverse Laplace transforms using the tables to find the current as a function of time. These tasks allow students understand how to use Laplace transforms for a time-domain solution for the current in the RLC circuit in response to the input voltage. This problem exemplifies how dual stance approach is used by employing the Laplace transform to convert a differential equations problem into an algebraic equations to manipulate and solve. This mathematical foundation is particularly useful in electrical engineering for analyzing circuits and systems' behavior in the time domain.

In Calculus sequences, we embedded assessments to measure and track students learning outcomes for solving differential equations including optimization problems, which prepare students solving engineering differential equations problems. From 2019 till 2023, our assessment results show that more than 70% consistently achieved this learning outcome.

B. Functional analysis in Introduction to Communication Theory and Systems

The first paragraph is edited to read as follows: " Introduction to Communication Theory and Systems is a core course for the development of our electrical engineering students. With an enrollment of twenty students every semester, it serves as a prerequisite course for signal processing and other upper-level classes. Students come to this class with mathematical gaps, addressed by this study, previously demanding the instructor and students spend a month to build their mathematical rigor.

The continuous and periodic aspects of transformations presented more earlier in calculus to build the mathematical foundations. It is essential for students to understand that Fourier transformation their which maps the continuous time domain to discrete frequency domain. This process of discretization is critical for digital signal processing or control systems. The student learning outcomes aligned with Functional Analysis learning trajectory include development of student knowledge of frequency domain and time domain response of linear systems; analog modulation methods including amplitude modulation, frequency modulation and phase modulation. ;The class starts with an introduction to communication systems

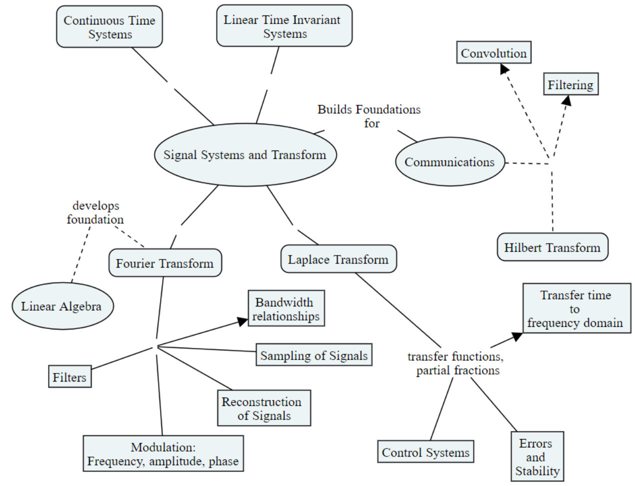

Fig. 4. Concept map depicting functional analysis learning trajectory within Signal Systems and Communication course

concepts such as mathematical models for communication channels.Students engage in learning tasks on signals and linear systems with an emphasis on signal transforms, Fourier methods, filter design, power, energy, the Hilbert transform and lowpass and bandpass signals. From those foundational concepts, students build modulation and demodulation techniques for amplitude, frequency and phase signals. The primary mathematical challenge within this communications course is that engineering concepts to be covered within a week takes about four weeks due to students' difficulties with the mathematical foundations. These gaps in students' mathematical background could have been addressed to support student success with a mathematically rigorous Signals, Systems and Transforms course as an upper-level class, which is not offered due to curricular constraints. This required us to develop curricular innovations with the existing engineering courses with strong mathematical supplements. In collaboration with engineering and mathematics faculty, the following concept map is developed as seen in Figure 4 breaking down the concepts in Signals, Systems, and Transforms related to functional analysis ideas to strengthen the students' understanding of classical and modern communication systems.

Functional analysis learning trajectory in this course focuses on students' learning to use the Fourier transform and the Laplace Transform to understand and study continuous time systems and linear time invariant systems (LTI). Those concepts also fortify other core classes such as linear circuits, electrical power, and control systems. Students are expected to develop and use Fourier Series as a foundation to transform a signal from time domain into frequency domain. However, series are not as critical in the application of signal modulation techniques as the Fourier transform. Students are engaged in learning task where a time signal becomes modulated and propagated through the spectrum in which noise and interference gets added. The receiving side is introduced to students as the design challenge. The Fourier transform enables students to break down the combination of sinusoidal waves into frequency harmonics. The student is then made aware how a signal of interest can be filtered through its detection in the frequency domain. The Fourier transform is introduced first with its integral form and then with its inverse form. Initially students use the calculus based integral of the transform to go from time domain into frequency domain for practice. Then, the students are tasked to solve the same problems utilizing a table of Fourier Transform Properties. Once the student can work on the properties, the implementation of the Fourier Transform integral is fully replaced with the Fourier Transform Pairs Table for its practicality. In this Signals, Systems and Transforms course, the engineering students learn to identify the Fourier transform applications to develop ideal filters, real filters, bandwidth relationships, sampling continuous-time signals, reconstructing signals, sinusoidal and pulse amplitude modulation. Once the students have that foundation, the Fourier transform is utilized in communications in terms of spectrum given certain Dirichlet conditions. In electrical engineering if critical to have a clear understanding of the fundamental sinusoidal equation: $Am\sin(\omega t \pm \theta)$ which defines the fundamental shape of a continuous electrical signal. The term ω is replaced by $2\pi f$ to enable the student to work the mathematical design problems in terms of frequency. Then, the relationship of radians in terms of ω to $2\pi f$ is expanded as $Am\sin(2\pi f \pm \theta)$. All Fourier transform properties and pairs are redefined in terms of frequency instead of radians. The first true implementation involves the study of amplitude modulation in which a message signal $m(t)$ is transmitted through a communication channel through a carrier signal: $c(t) = A_c \cos(2\pi f_c t + \theta_c)$ Then, students learn different methods of modulation utilizing the Fourier transform of a signal $u(t)$ to obtain the spectrum of a modulated signal:

$$u(t) = m(t)c(t) = A_c m(t)\cos(2\pi f_c t)$$

It is observed that the students are expected to have a good understanding of trigonometric functions to succesfully implement sinusoidal modeling in the study of communications, which aligns with the results in earlier research [9]. Engineering students are expected to obtain the spectra through Fourier methods to study the frequency and phase modulation. Bessel functions play a critical role in frequency modulation by determining the frequency components' magnitudes and signs in the frequency modulation (FM) spectrum. In FM, the modulating signal causes the carrier wave's frequency to vary, which generates sidebands around the carrier frequency. The number of sidebands and their amplitudes are determined by Bessel functions. Students are expected to utilize Bessel functions for frequency analysis and modulation. The understanding of phase modulation comes from the study of frequency modulation. From engineering stance, phase modulation is the foundation of study of digital signals in modern communications. To understand frequency and phase modulation, students learn to apply Fourier Transform, which serves as the most critical tool for their development into becoming an engineer.

Laplace Transform is the other important transform connecting the study of communications devices, circuitry, control theory, providing a practical method for solving differential equations. Electrical engineering student is expected to represent the flow of electric charge by a dynamic system explaining its behavior over time with differential equations. Laplace Transform assists simplifying this process by modifying all $\frac{d}{dt}$ terms by an $\mathbf{s}$ and all integral symbols by $\frac{1}{\mathbf{s}}$. Students are expected to understand and apply this transformation from time domain into the s-domain or plane. This becomes a critical skill in the study of transfer functions to identify systems dynamics and stability [26]. In the study of alternate current, as a consequence, $\mathbf{s}$ can be modified by $\mathbf{j}\omega$ in phasor analysis [27] involving complex analysis. Here, the Laplace Transform is initially studied by defining its integral definition $\mathcal{L}[f(t)] = F(s) = \int_{\infty}^{0} f(t)e^{-st}dt$

In communications Laplace transform is utilized as a tool to solve transfer functions and deal with Linear-Time Invariant Systems. Tables with Laplace Transform pairs and its properties are given to the students after the initial work is done with the integral to solve from time domain into the imaginary-real plane and vice versa through the inverse Laplace transform. However, the trough strength and benefit of the Laplace Transform has more meaning in the study of control theory.

C. Functional analysis in Control Systems

Building on the current local instructional practice, the concept map is collaboratively developed for the control course to articulate progression and connections of mathematical ideas related to functional analysis (Figure 5).

Students are required to model the motions for a mechanical system accounting for rotation and displacement with different degrees of freedom. Students study motions of a system composed of mass, spring, and damper system. Students initially draw a free body diagram. Based Newton's law to balance all forces in the system, students develop a second-degree ODE to study displacement over time. Based on the differential equation, Laplace transform is performed to convert ODE to an algebraic equation to solve in frequency domain. The inverse Laplace transform is used to find the solution of the system.

Changing the degree of freedom students then study more complex equations involving multiple variables such as using two masses, multiple strings, and damping conditions, towards modeling the displacement of multiple masses. With this extension, a linear system of simultaneous ordinary differential equations emerges, demanding strong mathematical foundation from students. The Laplace transform of the equation of motion is performed using appropriate transfer functions. Unlike previous case, this is not a simple algebraic equation but a linear system. Transfer functions are found by using Cramer's rule with the method of determinants. A solid foundation in linear algebra and ODEs are essential to solve the system of equations with determinants.

Students are then exposed to equations of motions problems with higher degrees of freedom, involving more than two

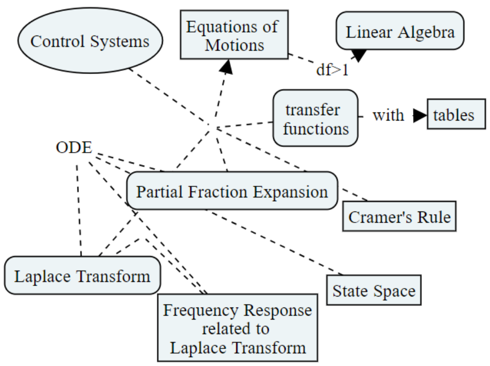

Fig. 5. Collaborative concept map for Control systems on Functional analysis learning trajectory

masses, demanding them to find the force required to be applied to a particular mass. The linear system now has a matrix of an equation involving as many equations as the number of masses to model their displacements. The end goal is to find the transfer function to find the relationship between the output (displacement) and input (force) of the system over time. Transfer functions for components are used to design, analyze and control the systems assembled from components. Transfer functions are derived from the Laplace transforms for input $\mathbf{x}(\mathbf{t})$ and output $\mathbf{y}(\mathbf{t})$.

In control class, state space models require students to represent a linear system with a mathematical model using first order differential equations to express current state to predict the future behavior of a system. State variables represented by the state vector $\mathbf{x}$ expressing the status of the storage elements such as inductor or capacitor in the system.

State and output equations are as follows;

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t); \mathbf{y}(t) = \mathbf{C}\mathbf{x}(t) + \mathbf{D}\mathbf{u}(t)$$

where the coefficients $\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D}$ are time-invariant or constant, representing state, input, output, and feedforward matrices. Eigenvalues of the matrix $\mathbf{A}$ are critical to determine the stability of the system. Students' strong background in linear algebra is critical to understand the stability of the system, including interpreting contextual meaning of different eigenvalues with real, complex, and repeating cases, with corresponding eigenvector. Students need to interpret the complex eigenvalues corresponding to oscillations around the directions of the corresponding eigenvectors by either increasing away from the origin or exponentially decreasing towards it. Students learn to interpret the contextual meanings of the signs of the real parts with negative real parts indicating stability. Students learn about the system's transfer functions to transform the differential equation into Laplace domain by taking the Laplace Transform of the 1st order ODE for the state to reduce it into a linear algebraic equation in frequency domain. Here the system's transfer function represents the ratio of Laplace transforms of the output and the input.

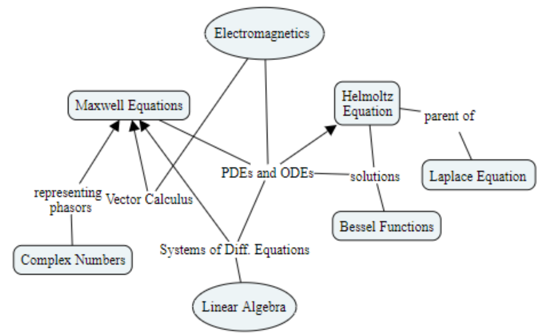

Fig. 6. Connections of Mathematical Demands in Electromagnetics for Engineering students related to Functional Analysis learning trajectory

With improved emphasis on Fourier series and transforms, and the Hilbert transform, there was a positive increase in the percent of students passing the class as well as demonstrating better understanding at the end of course assessments.

D. *Electromagnetics Theory*

Maxwell equations and Helmholtz equation are essential in building mathematical foundations in Electromagnetics, as seen in Figure 6. Students usually have hard time with Maxwell equations due to its complexity and its solution demands students to have some knowledge of functional analysis building on PDEs and integral transforms.

The connection provided here is that the solutions of the Helmholtz equation can be expanded into certain special "wave functions". The Laplacian is first represented in spherical coordinates and solutions are developed for the scalar Laplace equation or Helmholtz equation by separation of the variables. The spherical parts are eigensolutions of the Laplace–Beltrami operator. This operator is an extension of Laplace operator, like the Laplacian, is the (Riemannian) divergence of the (Riemannian) gradient, and it is a linear operator taking functions into functions. $\Delta f = \mathrm{div}(\nabla f)$ The radial part on the other hand solves an Euler type equation for the Laplace equation. The spherical Bessel differential equation is developed for the case of the Helmholtz equation. Their solutions are spherical harmonics and spherical Bessel and Hankel functions. Higher stance on Fourier transform is gained here with Hankel transform representing any given function as the weighted sum of an infinite number of Bessel functions of the first kind, connecting what they learned before with Bessel functions and ODEs. $\mathcal{F}_v(t) = \int_0^\infty f(x)\mathcal{J}_1(tx)x dx$ related to Fourier Bessel series in a finite interval. This is used to transform and solve Laplace equation in cylindrical coordinates.

We collected and compared the assessment results aligning with Accreditation Board for Engineering and Technology (ABET) reports, including on the learning outcome: "An ability to identify, formulate, and solve complex engineering problems by applying principles of engineering, science, and mathematics". ABET reports with specific performance indicators such as formative and summative assessments have been reported since 2017 in the specific EEEN 3320 Introduction to Communication Theory and Systems, and EEEN 3330 Control Theory I. That is in function to maintain the current ABET accreditation that the Electrical Engineering program currently has. There are also internal end of semester memos that help identify degrees of success and recommendations for future implementations teaching the course. We reflected on the assessment reports for these courses for prior years as we developed strategies inform functional analysis learning trajectories to map and address the gaps and targeted engineering mathematics connections represented by the collaborative concept maps. Based on the assessment results, students performed consistently above 80% between years 2017 and 2021 on the targeted learning outcome of analyzing frequency domain and time domain response of linear systems. By addressing these gaps in mathematical background for engineers, their performance reached 100 percent in years 2020 and 2021. The most challenging learning outcome was "design filters for signal noise removal in communication system applications" which demands additional mathematical interventions including statistics. This learning outcome was at the satisfactory level in year 2017 and reached to %80 in year 2021 meeting the target with the interventions.

V. Conclusion and Future Directions

Here we present the Functional analysis learning trajectory to design tasks to support its development across mathematical courses and mathematics intensive courses for engineering and applied mathematics education. The collaborative concept maps are developed by interdisciplinary engineering and mathematics faculty to support functional analysis learning trajectory for Calculus, ODE, Communications and Control systems courses process. These concept maps inform the instructional design of the dual modeling tasks sequences, establishing joint curricular innovation work among mathematics and engineering faculty to connect their courses . This research-to-practice study expands prior research [9] involving more engineering and mathematics faculty engaging in an interdisciplinary scholarship of teaching/learning mathematics to build cross curricular functional analysis learning trajectories. This work helps foster collaboration among faculty to design and implement dual perspective (engineering & Math) modeling task sequences across their courses and disciplines to support implementation of targeted learning trajectories. This orientation of research into practice exemplifies a research-based practice to build cross-curricular innovations by designing mathematical modeling task sequences along learning trajectories to build connections and progressions across mathematical ideas for engineers and mathematical scientists. An interdisciplinary scholarship of teaching and learning engineering mathematics education emerges among the engineering and mathematics faculty by collaborative concept mapping, the design and implementation of dual modeling task to support development of the Functional Analysis Learning trajectories in math and engineering courses(Calculus II, III, ODE, Communication, Control, electromagnetic) taken by engineering students.

This study informed our course based improvement efforts for curricular innovation for engineering mathematics education, including integration of FALT aligned assessments in math courses for engineering majors. These assessments are embedded into Calculus II and III, differential equations, and linear algebra to measure and track student learning progressions into targeted Engineering courses. We are currently experimenting with dual modeling task sequences with Jupyter notebooks and MATLAB live scripts for engineering and mathematics courses. Our future research includes the assessment and validation of the dual modeling task sequences refining the functional analysis learning trajectories. The validation studies will be conducted across courses incorporating with live notebooks supporting peer reflection, formative assessment. These live notebooks will help incorporate interdisciplinary mentoring for engineering and applied mathematics education to support students' development along the Functional Analysis Learning trajectory across courses.

We acknowledge that the curriculum and course organization in other institutions can be different. Therefore, more emphasis are given in the collaborative concept maps that are used to explicate functional analysis thinking across courses. This can be used as a tool for faculty and students to connect and build relevant ideas in their institution responding to the needs of their students and institutional constraints. We also used the ABET as a common framework adopted by institutions to guide and track learning outcome assessments to inform our course revisions.

Broader implications of this study include three emerging findings. The first is that more emphasis should be given on the mathematics courses on discrete functions and corresponding data modeling perspectives along the mathematics requirements. Duality perspective [28] between periodic and discrete functions emerges as the culminating higher stance to be gained for engineering mathematics education along the Functional analysis learning trajectory with Fourier methods. Fourier transform facilitates this duality as it maps the periodic functions into discrete functions, and the discrete functions into periodic functions. Our future research will focus on discretization process designing dual modeling tasks supporting students' understanding of discretization, and its applications in discrete time Fourier methods. During this study, complex analysis was found to be not adequately addressed in the mathematical requirements for engineering. One course was added as an elective to update the available course offerings and support engineering and applied mathematics education. Another major finding emerging from this study and demanded instructional practices is the necessity of integrating more rigorous integration of partial differential equations perspectives into calculus III and ODE to get students ready to tackle PDE problems emerging from engineering applications including Electromagnetism. The Fourier, Laplace methods, and series solutions demonstrated viable pathways for infusing PDE perspectives with the mathematical practices to convert PDEs into ODEs by separation of variables in different coordinate systems.

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